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On Integrodifferential Inequalities in Two Independent Variables

RAVI P. AGARWAL*

Mathematisches Institut der Ludwig-Maximilians-Universität, D-8000 Munich 2, West Germany

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Some new integrodifferential inequalities in two independent variables involving higher order partial derivatives have been obtained. Some applications are also given.

1. INTRODUCTION

In [11], we obtained several new integrodifferential inequalities involving higher order derivatives of single independent variable. These inequalities are directly useful in studying several properties of the solutions of ordinary differential equations, see also [12], where, essentially, these inequalities are used. The discrete inequalities with and without differences also share the same importance and are discussed in [1–3] and references therein. In the present paper we shall discuss several new integrodifferential inequalities in two independent variables involving higher order partial derivatives. Some particular cases of our results have been considered recently by Pachpatte [6–8], but in general the results obtained here cannot be compared with his results. It is also shown that his results can be improved uniformly. The results obtained here are useful in studying several properties of the solutions of a class of partial differential and integrodifferential equations.

2. LINEAR INEQUALITIES

LEMMA 1 [9, 10]. *Suppose $a(x, y)$ and $b(x, y) \geqslant 0$ are continuous functions on a domain D. Let $P_0(0, 0)$ and $P(x, y)$ be two points in D such that $xy > 0$ and let R be the rectangular region whose opposite corners are*

* Permanent address: Department of Mathematics, National University of Singapore, Kent Ridge, Singapore 0511.

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the points P_0 and P. Let $v(s,t,x,y)$ be the solution of the characteristic initial value problem

$$v_{st} - b(s,t)\,v = 0$$
$$v(s,y) = v(x,t) = 1$$

and let D^+ be a connected subdomain of D containing P such that $v \geqslant 0$ for all $(s,t) \in D^+$. If $R \subset D^+$ and

$$u_{xy} - b(x,y)\,u \leqslant a(x,y)$$
$$u(x,0) = u(0,y) = 0,$$

then

$$u(x,y) \leqslant \int_0^x \int_0^y a(s,t)\,v(s,t,x,y)\,dsdt.$$

LEMMA. [5]. *Let a, b, u, a_x, and a_{xy} be nonnegative continuous functions on R and*

$$u(x,y) \leqslant a(x,y) + \int_0^x \int_0^y b(s,t)\,u(s,t)\,dsdt. \tag{1}$$

Then

$$u(x,y) \leqslant a(x,y) \exp\left(\int_0^x \int_0^y b(s,t)\,dsdt\right). \tag{2}$$

This lemma is proved in [5] with no assumption on the sign of $a(x,y)$, and the term $a(x,y) \int_0^y b(x,t)\,dt \int_0^x b(s,y)\,ds$ is used as nonnegative.

Thus without $a(x,y) \geqslant 0$ their proof does not seem to be correct.

LEMMA 3. *In inequality* (1), *let a, b, and u be nonnegative continuous functions on R, and a be positive and nondecreasing. Then* (2) *holds.*

Let $a(x,y) = c(x) + d(y)$, where $c(x)$, $d(y) > 0$, and $c'(x)$, $d'(y) \geqslant 0$. Hence, conditions over $a(x,y)$ in both Lemmas 2 and 3 are satisfied and estimate (2) is a sharper estimate than obtained by Wendroff [4, p. 154].

THEOREM 4. *Suppose $a(x,y)$, $b(x,y)$, $h_{i,j}(x,y)$, and $\partial^{i+j}u(x,y)/\partial x^i \partial y^j$ $(0 \leqslant i \leqslant r_1,\ 0 \leqslant j \leqslant r_2)$ are nonnegative continuous functions on a domain D. Let $P_0(0,0)$ and $P(x,y)$ be two points in D such that $xy > 0$ and let R be the* *rectangular region whose opposite corners are the points P_0 and P. Let $v(s, t, x, y)$ be the solution of the characteristic initial value problem*

$$v_{st} - B_1(s, t)\, v = 0$$
$$v(x, y) = v(x, t) = 1$$

and let D^+ be a connected subdomain of D which contains P and on which $v \geqslant 0$. Then, if $R \subset D^+$ and

$$\frac{\partial^{r_1+r_2}u(x,y)}{\partial x^{r_1}\,\partial y^{r_2}} \leqslant a(x,y) + b(x,y) \sum_{i=0}^{r_1} \sum_{j=0}^{r_2} \int_0^x \int_0^y h_{i,j}(s,t) \times \frac{\partial^{i+j}u(s,t)}{\partial s^i\,\partial t^j}\, dsdt, \tag{3}$$

then

$$\frac{\partial^{r_1+r_2}u(x,y)}{\partial x^{r_1}\,\partial y^{r_2}} \leqslant a(x,y) + b(x,y) \int_0^x \int_0^y A_1(s,t)\, v(s,t,x,y)\, dsdt, \tag{4}$$

where

$$\begin{aligned} A_1(x,y) = {} & a(x,y)\, h_{r_1,r_2}(x,y) + \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} h_{i,j}(x,y) \\ & \times \Bigg[\sum_{p=j}^{r_2-1} \frac{y^{p-j}}{(p-j)!} \frac{\partial^{i+p}u(x,0)}{\partial x^i \partial y^p} + \sum_{q=i}^{r_1-1} \frac{x^{q-i}}{(q-i)!} \frac{\partial^{q+j}u(0,y)}{\partial x^q\,\partial y^j} \\ & - \sum_{p=j}^{r_2-1} \sum_{q=i}^{r_1-1} \frac{y^{p-j}}{(p-j)!} \frac{x^{q-i}}{(q-i)!} \frac{\partial^{q+p}u(0,0)}{\partial x^q \partial y^p} \\ & + \frac{1}{(r_1-i-1)!\,(r_2-j-1)!} \\ & \times \int_0^x \int_0^y (x-s)^{r_1-i-1}(y-t)^{r_2-j-1} a(s,t)\, dsdt \Bigg] \end{aligned} \tag{5}$$

$$\begin{aligned} B_1(x,y) = {} & h_{r_1,r_2}(x,y)\, b(x,y) + \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} h_{i,j}(x,y) \\ & \times \frac{1}{(r_1-i-1)!(r_2-j-1)!} \\ & \times \int_0^x \int_0^y (x-s)^{r_1-i-1}(y-t)^{r_2-j-1} b(s,t)\, dsdt \end{aligned} \tag{6}$$

$\alpha = r_1 - 1$, $\beta = r_2$ or $\alpha = r_1, \beta = r_2 - 1$. For $i = \alpha = r_1$ or $j = \beta = r_2$ the integrals in (5) *and* (6) *are interpreted in the usual way.*

Proof. Define a function $\phi(x, y)$ such that

$$\phi(x, y) = \sum_{i=0}^{r_1} \sum_{j=0}^{r_2} \int_0^x \int_0^y h_{i,j}(s, t) \frac{\partial^{i+j} u(s, t)}{\partial s^i \partial t^j} \, dsdt. \tag{7}$$

Then we have

$$\begin{aligned} \phi_{xy}(x, y) &= \sum_{i=0}^{r_1} \sum_{j=0}^{r_2} h_{i,j}(x, y) \frac{\partial^{i+j} u(x, y)}{\partial x^i \, \partial y^j} \\ \phi(0, y) &= \phi(x, 0) = 0. \end{aligned} \tag{8}$$

Using the definition of $\phi(x, y)$, (3) can be rewritten as

$$\frac{\partial^{r_1 + r_2} u(x, y)}{\partial x^{r_1} \, \partial y^{r_2}} \leqslant a(x, y) + b(x, y) \, \phi(x, y). \tag{9}$$

Integrating (9) $r_1 - i$ times with respect to x and $r_2 - j$ times with respect to y, it follows that

$$\begin{aligned} \frac{\partial^{i+j} u(x, y)}{\partial x^i \, \partial y^j} \leqslant & \sum_{p=j}^{r_2 - 1} \frac{y^{p-j}}{(p-j)!} \frac{\partial^{i+p} u(x, 0)}{\partial x^i \, \partial y^p} \\ & + \sum_{q=i}^{r_1 - 1} \frac{x^{q-i}}{(q-i)!} \frac{\partial^{q+j} u(0, y)}{\partial x^q \, \partial y^j} \\ & - \sum_{p=j}^{r_2 - 1} \sum_{q=i}^{r_1 - 1} \frac{y^{p-j}}{(p-j)!} \frac{x^{q-i}}{(q-i)!} \frac{\partial^{q+p} u(0, 0)}{\partial x^q \, \partial y^p} \\ & + \frac{1}{(r_1 - i - 1)! \, (r_2 - j - 1)!} \int_0^x \int_0^y (x - s)^{r_1 - i - 1} (y - t)^{r_2 - j - 1} \\ & \times [a(s, t) + b(s, t) \, \phi(s, t)] \, dsdt. \end{aligned} \tag{10}$$

Using (9) and (10) in (8), the nondecreasing nature of $\phi(x, y)$, and arranging the terms, we obtain

$$\phi_{xy}(x, y) \leqslant A_1(x, y) + B_1(x, y) \, \phi(x, y). \tag{11}$$

Now from Lemma 1, it follows that

$$\phi(x, y) \leqslant \int_0^x \int_0^y A_1(s, t) \, v(s, t, x, y) \, dsdt. \tag{12}$$

Substituting (12) in (9), result (4) follows. Note that from inequality (4), it follows that

$$\frac{\partial^{i+j}u(x,y)}{\partial x^i\,\partial y^j} \leqslant \sum_{p=j}^{r_2-1} \frac{y^{p-j}}{(p-j)!}\frac{\partial^{i+p}u(x,0)}{\partial x^i\,\partial y^p}$$
$$+ \sum_{q=i}^{r_1-1} \frac{x^{q-i}}{(q-i)!}\frac{\partial^{q+j}u(0,y)}{\partial x^q\,\partial y^j}$$
$$- \sum_{p=j}^{r_2-1}\sum_{q=i}^{r_1-1} \frac{y^{p-j}}{(p-j)!}\frac{x^{q-i}}{(q-i)!}\frac{\partial^{q+p}u(0,0)}{\partial x^q\,\partial y^p}$$
$$+ \frac{1}{(r_1-i-1)!\,(r_2-j-1)!}\int_0^x\int_0^y (x-s)^{r_1-i-1}(y-t)^{r_2-j-1}$$
$$\times \left[a(s,t)+b(s,t)\int_0^s\int_0^t A_1(\xi,\eta)\,v(\xi,\eta,s,t)\,d\xi d\eta\right] dsdt. \tag{13}$$

COROLLARY 5. *Suppose* $a(x,y)$, $b(x,y)$, $h_{i,j}(x,y)$, *and* $\partial^{i+j}u(x,y)/\partial x^i\partial y^j$ $(0 \leqslant i \leqslant r_1,\ 0 \leqslant j \leqslant r_2)$ *are nonnegative continuous functions on a domain* D, *and inequality* (3) *is satisfied. Then*

$$\frac{\partial^{r_1+r_2}u(x,y)}{\partial x^{r_1}\,\partial y^{r_2}} \leqslant a(x,y)+b(x,y)\left[\int_0^x\int_0^y A_1(s,t)\,dsdt\right.$$
$$\left.\times \exp\int_0^x\int_0^y B_1(s,t)\,dsdt\right]. \tag{14}$$

Proof. As in Theorem 4, it follows from (11) that

$$\phi(x,y) \leqslant \int_0^x\int_0^y A_1(s,t)\,dsdt + \int_0^x\int_0^y B_1(s,t)\,\phi(s,t)\,dsdt$$

and hence from Lemma 2, we find

$$\phi(x,y) \leqslant \left(\int_0^x\int_0^y A_1(s,t)\,dsdt\right)\exp\left(\int_0^x\int_0^y B_1(s,t)\,dsdt\right).$$

The result now follows from (9).

Remark 1. In inequality (3), let $r_2 = 0$ and y be fixed. Then the partial derivatives are considered as total derivatives and it is easy to verify that

$$A_1(x,y) = a(x,y)\,h_{r_1,0}(x,y) + \sum_{i=0}^{r_1-1} h_{i,0}(x,y)\left[\sum_{q=i}^{r_1-1}\frac{x^{q-i}}{(q-i)!}u^{(q)}(0,y)\right.$$
$$\left.+\frac{1}{(r_1-i-1)!}\int_0^x (x-s)^{r_1-i-1}\,a(s,y)\,ds\right]$$

$$B_1(x,y)=b(x,y)\,h_{r_1,0}(x,y)+\sum_{i=0}^{r_1-1} h_{i,0}(x,y)\left[\frac{1}{(r_1-i-1)!}\times\int_0^x (x-s)^{r_1-i-1}\,b(s,y)\,ds\right].$$

From (11), it follows that

$$\phi(x,y)\leqslant\int_0^x A_1(s,y)\exp\left(\int_s^x B_1(\tau,y)\,d\tau\right)ds$$

and hence

$$u^{(r_1)}(x,y)\leqslant a(x,y)+b(x,y)\int_0^x A_1(s,y)\times\exp\left(\int_s^x B_1(\tau,y)\,d\tau\right)ds. \tag{15}$$

Estimate (15) is the same as was obtained in [11, Theorem 1].

THEOREM 6. *Suppose $a(x,y)$, $b(x,y)$, $h(x,y)$, and $\partial^{2i}u(x,y)/\partial x^i\,\partial y^i$ $(0\leqslant i\leqslant r)$ are nonnegative continuous functions on a domain D. Let $P_0(0,0)$ and $P(x,y)$ be two points in D such that $xy>0$ and let R be the rectangular region whose opposite corners are the points P_0 and P. Let $v_i(s,t,x,y)$ and $(1\leqslant i\leqslant r+1)$ be the solutions of the characteristic initial value problems*

$$\begin{aligned}
&v_{1st}-[h(s,t)\,b(s,t)+h(s,t)+rb(s,t)+(r-1)]\,v_1=0\\
&v_{ist}-[h(s,t)\,b(s,t)+h(s,t)\\
&\qquad+(r-i+1)\,b(s,t)+(r-i-1)]\,v_i=0 \qquad (2\leqslant i\leqslant r)\\
&v_{r+1st}-[h(s,t)(b(s,t)-1)]\,v_{r+1}=0\\
&v_j(s,y)=v_j(x,t)=1 \qquad (1\leqslant j\leqslant r+1)
\end{aligned}$$

and let D^+ be a connected subdomain of D which contains P and on which $v_j\geqslant 0$ $(1\leqslant j\leqslant r+1)$. Then, if $R\subset D^+$ and

$$\frac{\partial^{2r}u(x,y)}{\partial x^r\,\partial y^r}\leqslant a(x,y)+b(x,y)\sum_{i=0}^{r}\int_0^x\int_0^y h(s,t)\frac{\partial^{2i}u(s,t)}{\partial s^i\,\partial t^i}\,dsdt, \tag{16}$$

then

$$\frac{\partial^{2r}u(x,y)}{\partial x^r\,\partial y^r}\leqslant a(x,y)+b(x,y)\int_0^x\int_0^y [H(x,y)+h(x,y)\,B_r(x,y)]\times v_{r+1}(s,t,x,y)\,dsdt, \tag{17}$$

where

$$B_1(x, y) = \int_0^x \int_0^y H(s, t)\, v_1(s, t, x, y)\, dsdt$$

$$B_i(x, y) = \int_0^x \int_0^y [H(s, t) + B_{i-1}(x, y)]\, v_i(s, t, x, y)\, dsdt \qquad (2 \leqslant i \leqslant r)$$

and

$$\begin{aligned} H(x, y) = h(x, y) \Bigg[a(x, y) + \sum_{i=0}^{r-1} \Bigg\{ &\sum_{p=i}^{r-1} \frac{y^{p-i}}{(p-i)!} \frac{\partial^{i+p} u(x, 0)}{\partial x^i\, \partial y^p} \\ &+ \sum_{q=i}^{r-1} \frac{x^{q-i}}{(q-i)!} \frac{\partial^{q+i} u(0, y)}{\partial x^q\, \partial y^i} \\ &- \sum_{p=i}^{r-1} \sum_{q=i}^{r-1} \frac{y^{p-i}}{(p-i)!} \frac{x^{q-i}}{(q-i)!} \frac{\partial^{q+p} u(0, 0)}{\partial x^q \partial y^p} \\ &+ \frac{1}{((r-i-1)!)^2} \int_0^x \int_0^y (x-s)^{r-i-1} (y-t)^{r-i-1} a(s, t)\, dsdt \Bigg\} \Bigg]. \end{aligned}$$

Proof. Let us define

$$\phi_1(x, y) = \sum_{i=0}^{r} \int_0^x \int_0^y h(s, t) \frac{\partial^{2i} u(s, t)}{\partial s^i\, \partial t^i}\, dsdt.$$

Then, it follows that

$$\phi_{1xy}(x, y) = h(x, y) \left[\frac{\partial^{2r} u(x, y)}{\partial x^r\, \partial y^r} + \sum_{i=0}^{r-1} \frac{\partial^{2i} u(x, y)}{\partial x^i\, \partial y^i} \right] \tag{18}$$

and from (16)

$$\frac{\partial^{2r} u(x, y)}{\partial x^r\, \partial y^r} \leqslant a(x, y) + b(x, y)\, \phi_1(x, y). \tag{19}$$

From (19) it is easy to verify that

$$\begin{aligned} \frac{\partial^{2i} u(x, y)}{\partial x^i\, \partial y^i} \leqslant &\sum_{p=i}^{r-1} \frac{y^{p-i}}{(p-i)!} \frac{\partial^{i+p} u(x, 0)}{\partial x^i\, \partial y^p} \\ &+ \sum_{q=i}^{r-1} \frac{x^{q-i}}{(q-i)!} \frac{\partial^{q+i} u(0, y)}{\partial x^q\, \partial y^i} \\ &- \sum_{p=i}^{r-1} \sum_{q=i}^{r-i} \frac{y^{p-i}}{(p-i)!} \frac{x^{q-i}}{(q-i)!} \frac{\partial^{q+p} u(0, 0)}{\partial x^q\, \partial y^p} \end{aligned}$$

$$+\frac{1}{((r-i-1)!)^2}\int_0^x\int_0^y (x-s)^{r-i-1}(y-t)^{r-i-1}$$
$$\times\,[a(s,t)+b(s\,t)\,\phi_1(s,t)]\,dsdt. \tag{20}$$

Using (19) and (20) in (18) and arranging the terms, we obtain

$$\phi_{1xy}(x,y)+h(x,y)\,\phi_1(x,y)\leqslant H(x,y)+h(x,y)\,b(x,y)\,\phi_1(x,y)$$
$$+\,h(x,y)\,\phi_2(x,y), \tag{21}$$

where

$$\phi_2(x,y)=\phi_1(x,y)+\sum_{i=0}^{r-1}\frac{1}{((r-i-1)!)^2}\int_0^x\int_0^y (x-s)^{r-i-1}(y-t)^{r-i-1}$$
$$\times\, b(s,t)\,\phi_1(s,t)\,dsdt.$$

From (21) and the fact that $\phi_1(x,y)\leqslant\phi_2(x,y)$, it follows that

$$\phi_{2xy}(x,y)+\phi_2(x,y)\leqslant H(x,y)+[h(x,y)\,b(x,y)+h(s,y)+b(x,y)]\,\phi_2(x,y)$$
$$+\,\phi_3(x,y),$$

where

$$\phi_3(x,y)=\phi_2(x,y)+\sum_{i=0}^{r-2}\frac{1}{((r-i-2)!)^2}\int_0^x\int_0^y (x-s)^{r-i-2}(y-t)^{r-i-2}b(s,t)$$
$$\times\,\phi_2(s,t)\,dsdt.$$

Once again using $\phi_2(x,y)\leqslant\phi_3(x,y)$, we find

$$\phi_{3xy}(x,y)+\phi_3(x,y)\leqslant H(x,y)+[h(x,y)\,b(x,y)+h(x,y)+2b(x,y)+1]$$
$$\times\,\phi_3(x,y)+\phi_4(x,y),$$

where

$$\phi_4(x,y)=\phi_3(x,y)+\sum_{i=0}^{r-3}\frac{1}{((r-i-3)!)^2}\int_0^x\int_0^y (x-s)^{r-i-3}(y-t)^{r-i-3}b(s,t)$$
$$\times\,\phi_3(s,t)\,dsdt.$$

Continuing in this way, we get

$$\phi_{rxy}(x,y)+\phi_r(x,y)\leqslant H(x,y)+[h(x,y)\,b(x,y)+h(x,y)$$
$$+\,(r-1)\,b(x,y)+(r-2)]$$
$$\times\,\phi_r(x,y)+\phi_{r+1}(x,y), \tag{22}$$

where

$$\phi_{r+1}(x, y) = \phi_r(x, y) + \int_0^x \int_0^y b(s, t)\, \phi_r(s, t)\, ds dt$$

and hence on using $\phi_r(x, y) \leqslant \phi_{r+1}(x, y)$

$$\phi_{r+1\; xy}(x, y) \leqslant H(x, y) + [h(x, y)\, b(x, y) + h(x, y) \\ + rb(x, y) + (r-1)]\, \phi_{r+1}(x, y),$$

where (note that) $\phi_j(x, 0) = \phi_j(0, y) = 0$, $1 \leqslant j \leqslant r+1$. Now from Lemma 1, it follows that

$$\phi_{r+1}(x, y) \leqslant \int_0^x \int_0^y H(s, t)\, v_1(s, t, x, y)\, ds dt = B_1(x, y). \tag{23}$$

Using (23) in (22) and again using Lemma 1, we find

$$\phi_r(x, y) \leqslant B_2(x, y).$$

Continuing this way, we obtain

$$\phi_2(x, y) \leqslant B_r(x, y). \tag{24}$$

Substituting (24) in (21), we get

$$\phi_{1xy}(x, y) - [h(x, y)(b(x, y) - 1)]\, \phi_1(x, y) \leqslant H(x, y) + h(x, y)\, B_r(x, y)$$

and hence from Lemma 1

$$\phi_1(x, y) \leqslant \int_0^x \int_0^y [H(s, t) + h(s, t)\, B_r(s, t)]\, v_{r+1}(s, t, x, y)\, ds dt$$

and now the result follows from (19).

Remark 2. A particular case of Theorem 6, $r = 1$ and $b = 1$, has been considered recently by Pachpatte [6, Theorem 5]. His estimate is sharper than obtained from our result; however, for this particular case even his estimate can be improved uniformly and this we shall consider in

THEOREM 7. *Suppose $a(x, y)$, $h(x, y)$, $u(x, y)$, and $u_{xy}(x, y)$ are nonnegative continuous functions on a domain D. Let $P_0(0, 0)$ and $P(x, y)$ be two points in D such that $xy > 0$ and let R be the rectangular region whose opposite corners are the points P_0 and P. Let $v(s, t, x, y)$ be the solution of the characteristic initial value problem*

$$v_{st} - [1 + h(s, t)]\, v = 0$$
$$v(s, y) = v(x, t) = 1$$

and let D^+ be a connected subdomain of D which contains P and on which $v \geqslant 0$. Then, if $R \subset D^+$ and $u(x, y)$ satisfies

$$u_{xy}(x, y) \leqslant a(x, y) + \int_0^x \int_0^y h(s, t)[u(s, t) + u_{st}(s, t)]\, dsdt, \tag{25}$$

then

$$\begin{aligned} u_{xy}(x, y) \leqslant a(x, y) + \int_0^x \int_0^y h(s, t)\Big[a(s, t) + H(s, t) \\ + \int_0^s \int_0^t \Big\{h(\xi, \eta)(a(\xi, \eta) + H(\xi, \eta)) \\ + a(\xi, \eta) - \int_0^\xi \int_0^\eta a(\alpha, \beta)\, d\alpha d\beta\Big\}\, v(\xi, \eta, s, t)\, d\xi d\eta\Big]\, dsdt. \end{aligned} \tag{26}$$

Proof. Define a function $\phi(x, y)$ such that

$$\phi(x, y) = \int_0^x \int_0^y h(s, t)[u(s, t) + u_{st}(s, t)]\, dsdt$$
$$\phi(0, y) = \phi(x, 0) = 0.$$

We then have

$$\phi_{xy}(x, y) = h(x, y)[u(x, y) + u_{xy}(x, y)]. \tag{27}$$

Using the definition of $\phi(x, y)$, (25) can be rewritten as

$$u_{xy}(x, y) \leqslant a(x, y) + \phi(x, y) \tag{28}$$

and hence

$$u(x, y) \leqslant H(x, y) + \int_0^x \int_0^y [a(s, t) + \phi(s, t)]\, dsdt, \tag{29}$$

where $H(x, y) = u(x, 0) + u(0, y) - u(0, 0)$. Using (28) and (29) in (27), we have

$$\begin{aligned} \phi_{xy}(x, y) \leqslant h(x, y) \Big[a(x, y) + H(s, y) + \phi(x, y) \\ + \int_0^x \int_0^y [a(s, t) + \phi(s, t)]\, dsdt\Big]. \end{aligned} \tag{30}$$

Define

$$\begin{aligned} \psi(x,y) &= \phi(x,y) + \int_0^x \int_0^y [a(s,t) + \phi(s,t)]\, dsdt \\ \psi(0,y) &= \psi(x,0) = 0. \end{aligned} \tag{31}$$

We then obtain

$$\psi_{xy}(x,y) = \phi_{xy}(x,y) + a(x,y) + \phi(x,y)$$

which is, from (30) and (31),

$$\psi_{xy}(x,y) \leqslant h(x,y)[a(x,y) + H(x,y) + \psi(x,y)] + a(x,y) \\ + \psi(x,y) - \int_0^x \int_0^y a(s,t)\, dsdt.$$

Now using Lemma 1, it follows that

$$\psi(x,y) \leqslant \int_0^x \int_0^y [h(s,t)(a(s,t) + H(s,t)) + a(s,t) \\ - \int_0^s \int_0^t a(\xi,\eta)\, d\xi d\eta]\, v(s,t,x,y)\, dsdt.$$

Substituting this estimate in (30) and integrating, we obtain

$$\begin{aligned} \phi(x,y) \leqslant \int_0^x \int_0^y h(s,t)[a(s,t) + H(s,t) \\ + \int_0^s \int_0^t \{h(\xi,\eta)(a(\xi,\eta) + H(\xi,\eta)) + a(\xi,\eta) \\ - \int_0^\xi \int_0^\eta a(\alpha,\beta)\, d\alpha d\beta\}\, v(\xi,\eta,s,t)\, d\xi d\eta]\, dsdt. \end{aligned}$$

Now the result follows from (28).

THEOREM 8. *Let $h(x,y)$, $\partial^{2i}u(x,y)/\partial x^i\,\partial y^i$ $(0 \leqslant i \leqslant r)$ be nonnegative continuous functions defined for $x \geqslant 0, y \geqslant 0$, and $\partial^i u(x,0)/\partial x^i = \partial^i u(0,y)/\partial y^i = 0$ $(0 \leqslant i \leqslant r-1)$ for which the inequality*

$$\frac{\partial^{2r}u(x,y)}{\partial x^r\,\partial y^r} \leqslant a(x) + b(y) + \sum_{i=0}^{r} \int_0^x \int_0^y h(s,t) \frac{\partial^{2i}u(s,t)}{\partial s^i\,\partial t^i}\, dsdt \tag{32}$$

holds for $x \geqslant 0$ and $y \geqslant 0$, where $a(x)$, $b(y)$, $a'(x)$, and $b'(y)$ are nonnegative continuous functions for $x \geqslant 0$ and $y \geqslant 0$. Then

$$\frac{\partial^{2r} u(x,y)}{\partial x^r \, \partial y^r} \leqslant B_{r+1}(x,y), \tag{33}$$

where

$$B_1(x,y) = [a(x)+b(y)] \exp\left(\int_0^x \int_0^y (h(s,t)+r)\, ds dt\right) \tag{34}$$

$$B_i(x,y) = [a(x)+b(y)] + \int_0^x \int_0^y (h(s,t)+(r-i+1))\, B_{i-1}(s,t)\, ds dt \quad (2 \leqslant i \leqslant r+1). \tag{35}$$

Proof. Define a function $\phi_1(x,y)$ by the right side of (32). Then as in Theorem 6, it follows that

$$\phi_{1xy}(x,y) \leqslant h(x,y)\, \phi_2(x,y), \tag{36}$$

where

$$\phi_2(x,y) = \phi_1(x,y) + \sum_{i=0}^{r-1} \frac{1}{((r-i-1)!)^2} \int_0^x \int_0^y (x-s)^{r-i-1} (y-t)^{r-i-1} \times \phi_1(s,t)\, ds dt$$

and hence on using $\phi_1(x,y) \leqslant \phi_2(x,y)$, we find

$$\phi_{2xy}(x,y) \leqslant h(x,y)\, \phi_2(x,y) + \phi_3(x,y), \tag{37}$$

where

$$\phi_3(x,y) = \phi_2(x,y) + \sum_{i=0}^{r-2} \frac{1}{((r-i-2)!)^2} \int_0^x \int_0^y (x-s)^{r-i-2} (y-t)^{r-i-2} \times \phi_2(s,t)\, ds dt.$$

Again using $\phi_2(x,y) \leqslant \phi_3(x,y)$, we get

$$\phi_{3xy}(x,y) \leqslant [h(x,y)+1]\, \phi_3(x,y) + \phi_4(x,y). \tag{38}$$

Continuing this way, we obtain

$$\phi_{rxy}(x,y) \leqslant [h(x,y)+(r-2)]\, \phi_r(x,y) + \phi_{r+1}(x,y), \tag{39}$$

where

$$\phi_{r+1}(x, y) = \phi_r(x, y) + \int_0^x \int_0^y \phi_r(s, t)\, dsdt$$

and hence on using $\phi_r(x, y) \leqslant \phi_{r+1}(x, y)$ to obtain

$$\begin{aligned}\phi_{r+1xy}(x, y) &\leqslant [h(x, y) + (r-2)]\, \phi_{r+1}(x, y) + \phi_{r+1}(x, y) + \phi_{r+1}(x, y)\\ &= [h(x, y) + r]\, \phi_{r+1}(x, y).\end{aligned} \tag{40}$$

From the definition of $\phi_i(x, y)$ it follows that $\phi_i(x, y) \leqslant \phi_{i+1}(x, y)$, $1 \leqslant i \leqslant r$, and $\phi_j(x, 0) = a(x) + b(0)$, $\phi_j(0, y) = a(0) + b(y)$, $1 \leqslant j \leqslant r+1$. Integrating both sides of (40), we find

$$\phi_{r+1}(x, y) \leqslant a(x) + b(y) + \int_0^x \int_0^y [h(s, t) + r]\, \phi_{r+1}(s, t)\, dsdt. \tag{41}$$

Now an application of Lemma 2 gives

$$\phi_{r+1}(x, y) \leqslant B_1(x, y). \tag{42}$$

Substituting (42) in (39) and using $\phi_r(x, y) \leqslant \phi_{r+1}(x, y)$, we find

$$\phi_{rxy}(x, y) \leqslant [h(x, y) + (r-1)]\, B_1(x, y)$$

and hence

$$\phi_r(x, y) \leqslant B_2(x, y).$$

Continuing this way result (33) follows.

Remark 3. A particular case of Theorem 8, $r = 1$, has been considered by Pachpatte [7, Theorem 1] under more strong conditions (also, the estimate obtained here is sharper than his result).

3. Some Applications

Here we shall present some applications to our results obtained in Section 2. Note that in Theorem 4, $B_1(x, y)$ is nonnegative and the function $v(s, t, x, y)$ is the well-known Riemann function relative to the point $P(x, y)$. The existence, continuity, and nonnegative properties are well known; e.g., see [9, 10].

We shall consider the nonlinear hyperbolic partial integrodifferential equation

$$\frac{\partial^{r_1+r_2}u(x,y)}{\partial x^{r_1}\partial y^{r_2}} = a(x,y) + \int_0^x \int_0^y f\left(x, y, s, t, u(s,t), \frac{\partial u(s,t)}{\partial s}, \ldots, \right.$$
$$\left. \times \frac{\partial^{i+j}u(s,t)}{\partial s^i \partial t^j}, \ldots, \frac{\partial^{r_1+r_2}u(s,t)}{\partial s^{r_1}\partial t^{r_2}}\right) dsdt,$$
$$0 \leqslant i \leqslant r_1, \qquad 0 \leqslant j \leqslant r_2 \quad (43)$$

together with prescribed boundary conditions $\partial^i u(x,0)/\partial x^i$ and $\partial^j u(0,y)/\partial y^j$. The functions a and f are assumed to be continuous in all their arguments.

EXAMPLE 1. (Uniqueness). On the domain of definition of f let the following Lipschitz condition be satisfied:

$$\left| f\left(x, y, s, t, u(s,t), \frac{\partial u(s,t)}{\partial s}, \ldots, \frac{\partial^{r_1+r_2}u(s,t)}{\partial s^{r_1}\partial t^{r_2}}\right) \right.$$
$$\left. - f\left(x, y, s, t, \bar{u}(s,t), \frac{\partial \bar{u}(s,t)}{\partial s\, \partial t}, \ldots, \frac{\partial^{r_1+r_2}\bar{u}(s,t)}{\partial s^{r_1}\partial t^{r_2}}\right) \right|$$
$$\leqslant b(x,y) \sum_{i=0}^{r_1} \sum_{j=0}^{r_2} h_{i,j}(s,t) \left| \frac{\partial^{i+j}u(s,t)}{\partial s^i \partial t^j} - \frac{\partial^{i+j}\bar{u}(s,t)}{\partial s^i \partial t^j} \right|, \qquad (44)$$

where $b(x,y)$ and $h_{i,j}(s,t)$ are nonnegative continuous functions on D. Let us assume that there are two solutions $u_1(x,y)$ and $u_2(x,y)$ of (43) satisfying the prescribed boundary conditions. Then, from (44) it follows that

$$\left| \frac{\partial^{r_1+r_2}\phi(x,y)}{\partial x^{r_1}\partial y^{r_2}} \right| \leqslant b(x,y) \sum_{i=0}^{r_1} \sum_{j=0}^{r_2} \int_0^x \int_0^y h_{i,j}(s,t) \left| \frac{\partial^{i+j}\phi(s,t)}{\partial s^i \partial t^j} \right| dsdt, \qquad (45)$$

where $\phi(x,y) = u_1(x,y) - u_2(x,y)$. Define

$$\psi(x,y) = \sum_{i=0}^{r_1} \sum_{j=0}^{r_2} \int_0^x \int_0^y h_{i,j}(s,t) \left| \frac{\partial^{i+j}\phi(s,t)}{\partial s^i \partial t^j} \right| dsdt.$$

Then, from (45),

$$\left| \frac{\partial^{r_1+r_2}\phi(x,y)}{\partial x^{r_1}\partial y^{r_2}} \right| \leqslant b(x,y)\, \psi(x,y). \qquad (46)$$

Since

$$\frac{\partial^{i+j}\phi(x,0)}{\partial x^i \partial y^j} = \frac{\partial^{i+j}\phi(0,y)}{\partial x^i \partial y^j} = 0, \qquad 0 \leqslant i \leqslant r_1, \text{ and } 0 \leqslant j \leqslant r_2,$$

it follows as in Theorem 4 that

$$\left|\frac{\partial^{i+j}\phi(x,y)}{\partial x^i\,\partial y^j}\right| \leqslant \frac{1}{(r_1-i-1)!\,(r_2-j-1)!} \times \int_0^x \int_0^y (x-s)^{r_1-i-1}(y-t)^{r_2-j-1} b(s,t)\,\psi(s,t)\,dsdt.$$

Thus, we find

$$\psi_{xy}(x,y) \leqslant B_1(x,y)\,\psi(x,y)$$

and hence as in Theorem 4, we get $|\partial^{r_1+r_2}\phi(x,y)/\partial x^{r_1}\,\partial y^{r_2}| \leqslant 0$ which implies $|\phi(x,y)| \leqslant 0$ and thus $u_1(x,y) \equiv u_2(x,y)$.

EXAMPLE 2 (Upper Bound). Consider (43) with the boundary conditions

$$\frac{\partial^i u(x,0)}{\partial x^i} = \frac{\partial^j u(0,y)}{\partial y^j} = 0, \qquad 0 \leqslant i \leqslant r_1 - 1, \quad 0 \leqslant j \leqslant r_2 - 1$$

and

$$\left| f\left(x, y, s, t, u(s,t), \frac{\partial u(s,t)}{\partial s}, \ldots, \frac{\partial^{r_1+r_2}u(s,t)}{\partial s^{r_1}\,\partial t^{r_2}}\right)\right| \leqslant b(x,y) \sum_{i=0}^{r_1} \sum_{j=0}^{r_2} h_{i,j}(s,t) \left|\frac{\partial^{i+j}u(s,t)}{\partial s^i\,\partial t^j}\right|.$$

If $u(x,y)$ is any solution of (43), then it follows that

$$\left|\frac{\partial^{r_1+r_2}u(x,y)}{\partial x^{r_1}\,\partial y^{r_2}}\right| \leqslant |a(x,y)| + b(x,y) \sum_{i=0}^{r_1} \sum_{j=0}^{r_2} \int_0^x \int_0^y h_{i,j}(s,t) \times \left|\frac{\partial^{i+j}u(s,t)}{\partial s^i\,\partial t^j}\right| dsdt. \tag{47}$$

Now, following (as in Example 1) an application of Corollary 5,

$$\left|\frac{\partial^{r_1+r_2}u(x,y)}{\partial x^1\,\partial y^{r_2}}\right| \leqslant |a(x,y)| + b(x,y) \left[\int_0^x \int_0^y A_1^*(s,t)\,dsdt \times \exp \int_0^x \int_0^y B_1(s,t)\,dsdt\right], \tag{48}$$

where

$$A_1^*(x, y) = |a(x, y)|\, h_{r_1, r_2}(x, y) + \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} h_{i,j}(x, y) \frac{1}{(r_1 - i - 1)!\,(r_2 - j - 1)!}$$

$$\times \int_0^x \int_0^y (x-s)^{r_1 - i - 1}(y-t)^{r_2 - j - 1} |a(s, t)|\, ds dt.$$

Consider once again (43) with $r_1 = r_2 = r$ and

$$\frac{\partial^i u(x, 0)}{\partial x^i} = \frac{\partial^i u(0, y)}{\partial y^i} = 0, \qquad 0 \leqslant i \leqslant r - 1,$$

and

$$\left| f\left(x, y, s, t, u(s, t), \frac{\partial u(s, t)}{\partial s}, \ldots, \frac{\partial^{2r} u(s, t)}{\partial s^r\, \partial t^r}\right) \right|$$

$$\leqslant \sum_{i=0}^{r} h(s, t) \left| \frac{\partial^{2i} u(s, t)}{\partial s^i\, \partial t^i} \right|$$

and, also

$$|a(x, y)| \leqslant M$$

$$\left| \frac{\partial^{2r} u(x, y)}{\partial x^r\, \partial y^r} \right| \leqslant M + \sum_{i=0}^{r} \int_0^x \int_0^y h(s, t) \left| \frac{\partial^{2i} u(s, t)}{\partial s^i\, \partial t^i} \right| ds dt.$$

An application of Theorem 8 gives

$$\left| \frac{\partial^{2r} u(x, y)}{\partial x^r\, \partial y^r} \right| \leqslant B_{r+1}^*(x, y), \tag{49}$$

where

$$B_1^*(x, y) = M \exp\left(\int_0^x \int_0^y (h(s, t) + r)\, ds dt \right)$$

$$B_i^*(x, y) = M + \int_0^x \int_0^y (h(s, t) + (r - i + 1))\, B_{i-1}^*(s, t)\, ds dt$$

$$(2 \leqslant i \leqslant r + 1).$$

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